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International Journal of Heat and Mass Transfer 50 (2007) 1675–1682
International Journal of HEAT and MASS TRANSFER
www.elsevier.com/locate/ijhmt

Optimization of heat conduction using combinatorial optimization algorithms

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Received 8 April 2005; received in revised form 21 August 2006
Available online 11 January 2007

Abstract

The Volume-to-Point (VP) problem is a base problem of heat conduction optimization. The nonlinear two-dimensional optimization problem of VP is discretized and transformed to a combinatorial optimization problem, which can be solved by some modern optimization algorithms. Algorithms for VP problem using simulated annealing and genetic algorithm are developed. Results for different cases are obtained using these algorithms. Analyses of the results and algorithms are also presented, that shows these algorithms are better than bionic optimization algorithm and constructal theory for VP problem, and can be generalized to complex conditions.

Keywords: Heat conduction; Volume-to-Point; Optimization; Simulated annealing; Genetic algorithm

1. Introduction

Volume-to-Point (VP) or Area-to-Point heat conduction problem was originally defined by Bejan [1,2]: Consider a finite volume with uniform heat source inside, which is cooled through a small patch of heat sink located on its boundary. A finite amount of high conductivity material can be inserted to the volume. Determine the optimal distribution of high conductivity material through the given volume such that the highest temperature is minimized.

The VP problem is a fundamental problem of cooling for electronics. With increasing of power, as well as decreasing size of electronics, the cooling problem becomes increasingly important. The heat generated inside of electronics must be ejected outside quickly or the temperature will exceed the limit. Due to the small size, convective cooling method is impractical because the ducts take too much space. The problem can be overcome by construction of high efficient conductive path in the electronics. By insertion of high conductivity material such as diamond or carbon fiber the heat inside can be ejected more effectively. How to construct the heat-transport path using a limited amount of high conductivity material to minimize the highest temperature is an optimization problem. VP problems can also be found in many engineering fields, such as heating or cooling chemical reaction, cooling food [3] and increasing the thermal conductivity of energy storage media by carbon fibers [4]. Further more, VP problem has applications in a wide range of dissimilar sciences such as biology, economics, urban transportation, etc. [2,5–8].

In general, VP problem is a geometric optimization to minimize the flow resistance of a volume. To solve VP problem, Bejan [1] presented tree-network constructs based on constructal theory. The process of constructal theory starts up with an optimal element area, and then assembles the areas to a larger area. The heat-transport path got by constructal theory is like a tree with several levels of branches, in which number and size of branches of each level are optimized to minimize highest temperature.

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doi:10.1016/j.ijheatmasstransfer.2006.10.037

Nomenclature

A	coefficient matrix
b	vector of source term
$\hat{k}$	ratio of k_p and k_0
K	number
k	thermal conductivity
k_0	thermal conductivity of base material
k_p	thermal conductivity of high conductivity material
L	length of square area
M	population
m	number of high conductivity element
N	number of total elements
n	grid number of length
P	distribution of high conductivity material
p	material type of element
q	heat generation rate per unit volume
T	temperature
t	annealing temperature
T_m	maximum temperature
T_s	temperature of sink
V	total volume
V_p	volume of high conductivity material
x, y	Cartesian coordinates

Greek symbols

α	factor of annealing
ΔT	temperature difference
ϕ	filling ratio
ρ	nondimensional thermal resistance

Subscripts

i	index of element
k	step number of annealing

Constructal theory has many successful applications in heat conduction, flow, economics and other fields in both engineering and nature [2]. In Ref. [2], Bejan also presented several other algorithms to solve VP problem. The constructal theory is based on assumptions that $k_p \kappa k_0$ and the volume fraction of high conductivity material is small. Ghodoossi [8,9] indicated that higher order assembly dose not always improve the performance, for $\hat{k}\phi < 8$, the element area is better than high order assembly, and for $\hat{k}\phi > 8$, the first order assembly is the best.

Xia [10], Guo [11] and Cheng [12] developed bionic optimization algorithm to solve VP problem. The principle used in the algorithm indicates that the mean temperature is lowest when the temperature gradient distribution of the volume is most uniform. The bionic optimization imitates the biological evolution in nature, which consists of a generation process and a degeneration process. In each step of the evolution process, high conductivity material is shifted to the position with highest temperature gradient from the position with lowest temperature gradient. The mean temperature is getting lower while the evolution going on. The "best" distribution is got when the mean temperature is not lower anymore. Bionic optimization can get better solution than constructal theory [12].

The objective of bionic optimization algorithm is to minimize the mean temperature, but not the highest temperature. Simulations by authors of this article show that results of bionic optimization rely on the initial distribution of high conductivity material, and which initial distribution can get the best solution is unpredictable. So what is the best solution of VP problem? This article will try to solve it in another way. The original VP problem is discretized and transformed to a normal combinatorial optimization problem, which can be solved by modern optimization algorithms such as random local search, simulated annealing and genetic algorithm. The later two algorithms are developed in this article for VP problem, and are used to solve some examples.

2. Math models

The 2D VP problem is shown in Fig. 1. The volume is square, which is adiabatic on the boundary except the heat sink T_s located on the middle of bottom boundary. The width of heat sink δ is much less than L. The volume has uniform heat source q. The volume of k_p material V_p is much smaller than the total volume V. The filling ratio is defined as $\phi = V_p/V$, about 2–15%.

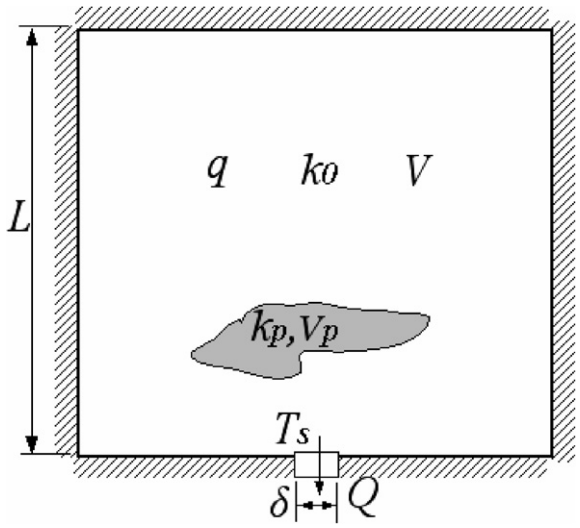

Fig. 1. 2D Volume-to-Point problem.

The math model of the VP problem is:

$$\min(\max T) \tag{1}$$

$$\text{s.t. } \nabla(k\nabla T) + q = 0 \tag{2}$$

$$\begin{cases} T = T_s & -\delta/2 < x < \delta/2; y = 0 \\ \partial T/\partial x = 0 & x = -L/2, L/2 \\ \partial T/\partial y = 0 & y = 0, L; x < -\delta/2, x > \delta/2 \end{cases} \tag{3}$$

$$\int_{V_p} \mathrm{d}V = \phi V \tag{4}$$

Eq. (1) is the objective and Eqs. (2)–(4) are the constraint conditions. Eq. (2) is the energy conservation equation, in which k is different for base material and high conductivity material; Eq. (3) is the boundary conditions of the energy conservation equation; Eq. (4) is the constraint for high conductivity material.

The optimization problem of VP is difficult to solve because it is a nonlinear optimization problem. Eq. (2) cannot be solved analytically, while it can be solved by numerical method such as finite difference and finite volume method. We can divide the volume into small cells by $n \times n$ grid uniformly, shown in Fig. 2, and solve the temperature field using finite volume method. Each cell is an element with determinate conductivity and uniform temperature. Then Eq. (2) is transformed to a group of linear algebraic equations [13]:

$$AT = b \tag{5}$$

where A is the coefficient matrix depended on the distribution of conductivity; b is a vector of source term related to q, and T is the temperature vector of all elements. Eq. (7) can be solved effectively using Cholesky factorization [14] because A is a positive definite, symmetric, sparse matrix.

Now the VP optimization problem becomes a linear optimization problem:

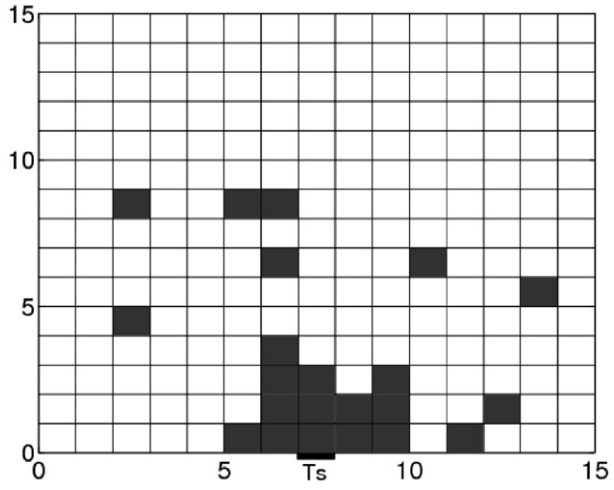

Fig. 2. Discrete Volume-to-Point problem.

$$\min(\max T_i) \tag{6}$$

$$\text{s.t. } A(P)T = b \tag{7}$$

$$\sum_i p_i = m, \quad p_i = 0, 1 \tag{8}$$

where $P = \{p_i | i = 1, N\}$ stands for distribution of high conductivity material, $m = \phi N$, is the number of k_p elements. If $p_i = 1$, conductivity of element i is k_p, otherwise k_0. A depends on P, and each combination of P is a solution of the problem. Given a combination of P, the temperature distribution T can be solved, and the maximum temperature is got. So the new optimization problem is a combinatorial optimization problem. This combinatorial optimization problem seems like a NPC or NP-hard problem [15] because the combination of P is $\binom{N}{m}$. It is difficult to find a polynomial algorithm for this problem to get the best solution, but the approximate best solution can be got by some modern algorithms such as simulated annealing [16] and genetic algorithm [17].

3. Algorithms

3.1. Simulated Annealing

The Simulated Annealing (SA) exploits an analogy between the way in which a metal cools and freezes into a minimum energy crystalline structure (the annealing process) and the search for a minimum in a more general system [16]. In the optimization process, the solution randomly walks in its neighborhood with a probability determined by Metropolis principle [18] while the system temperature decreases slowly; when the annealing temperature is closing zero, the solution stays at the global best solution in a high probability.

The procedure of SA for VP problem is:

(1) randomly choose an initial solution P, calculate its maximum temperature T_m; set system temperature $t := t_0$;
(2) do the random walk for K loops: randomly choose a neighbor of P, P' ; calculate maximum temperature of P', T'_m;
if $T_m > T'_m$ **or** $\exp(-(T'_m > T_m)/t) > \text{random}[0, 1)$
$P = P', T_m = T'_m$
(3) if terminating condition is matched, stop; else, annealing: $t := \alpha t$, go to (2)

where t is the system annealing temperature which has no relations to temperature T. The initial annealing temperature t_0 can be set as the volume's mean temperature without high conductivity material, because at this temperature almost all walk of solution can be accepted. K should be several times of N, random $[0, 1)$ is a random number less than 1 and no less than 0, generated by an uniform distribution random number generator. The annealing schedule is

$t_{k+1} = \alpha t_k$, where $0.85 < \alpha < 0.95$. The terminating condition is: if P has not changed for dozens of loops, stop annealing and return P as the best solution.

The solutions got by SA are approximate to the global best solution, but seldom equal to it exactly for large scale problem ($N > 50$). The performance of SA depends on the algorithm of neighbor selection. The algorithm used in this article is: randomly chooses an element of k_p, exchanges its conductivity with an element of k_0 randomly chosen, and the distance between the two elements is randomly chosen by a normal (Gauss) distribution random number generator. To improve the performance, tempering–annealing algorithm [19] is used: when solution has not changed for dozens of times, increases annealing temperature t then do annealing procedure again; repeat tempering–annealing for several times and return the best solution.

3.2. Genetic algorithm

The genetic algorithm (GA) is based on the Darwinian theory of nature selection and survival of fittest that exist in the genetics of the species [17]. To use GA in VP problem, some techniques such as fitness, encoding of chromosome, crossover operator are implemented as:

- The sorting fitness function is used: sort the population by their maximum temperature in descending order, randomly select i individual with the probability $2i/(M(M+1))$, where M is the population.
- Encoding of chromosome: chromosome is the distribution of k_p material; we record the indexes of k_p elements to a vector, such as [3, 7, 15, 34, 55, ...], as the chromosome.
- Crossover operator: select 2 chromosomes from reproduction, randomly change their genes (indexes of k_p elements) to product 2 children.
- Mutation operator: randomly change a gene of the chromosome.

The GA's procedure is:

(1) randomly choose M distributions of k_p as the initial population;
(2) calculate the maximum temperature of each chromosome;
(3) sort the population and generate reproduction;
(4) do crossover for every 2 of reproduction, product the children population;
(5) randomly choose a chromosome from children population, do mutation;
(6) if terminating condition is matched, stop; else go to (2).

The population M can be several times of elements, and terminating condition is the same as that of SA. During the GA's procedure, the best 2 individuals of each generation are saved and delivered to next generation without changes.

GA can get the global best solution theoretically, but it can only get approximate best solution due to the limitation of population scale.

4. Results and analysis

The optimized results of VP problems depend on grid number, δ, $\hat{k}$ and ϕ, but have no relations to L, k_0, q and T_s. To compare the results with that by constructal theory and bionic optimization algorithm (BO), the nondimensional thermal resistance of overall is defined referring to Ref. [1]:

$$\rho = \frac{\Delta T_{\max}}{qL^2/k_0} \tag{9}$$

ρ is determined by distribution of k_p material, the better distribution, the lower ρ.

4.1. Compare with bionic optimization algorithm

Several cases of different grid numbers: 21×21, 51×51, 101×101 and different $\hat{k}$: 3, 10, 100 are solved. In these cases the width of heat sink is one element's length and ϕ is 0.1. These cases are also solved by bionic optimization algorithm, so that the results can be compared.

The distribution of k_p material got by SA/GA algorithms for 21×21, $\hat{k} = 3, 10, 100$ are shown in Fig. 3.

Fig. 4 is the results for 51×51, $\hat{k} = 3, 10, 100$ by SA.

Fig. 5 is the results for 101×101, $\hat{k} = 3, 10, 100$ by SA. The results got by BO for the same cases are shown in Fig. 6.

From these results, we get the shape's characteristics of high conductivity material of VP problem: All high conductivity material is continuous and there are no holes in it; For cases with low $\hat{k}$, the shape of high conductivity material is thick and short, surrounding the heat sink; As increasing $\hat{k}$, the shape becomes slender, and extends to the corner far from the heat sink.

The shapes of k_p material got by combinatorial optimization algorithms are similar to the simulated structures of river drainage basins presented in Chapter 6 of Ref. [2]. The river drainage basin is also a kind of volume-to-point dissimilar problem, which is an optimization problem to minimize the overall resistance by arranging the internal structure. The algorithm used to get river drainage network in Ref. [2] is an evolutionary process: in each step, increases flow-rate parameter M from zero to critical value M_c, then replaces the dislodged blocks of k_0 material with k_p material. The algorithm is different from the combinatorial optimization algorithm in this article.

The nondimensional thermal resistance of these cases are listed in Table 1, where the third row is the difference of the two algorithms. For low $\hat{k}$, the difference between SA/GA and BO is small, while for high $\hat{k}$, results got by SA/GA are much better than that by BO.

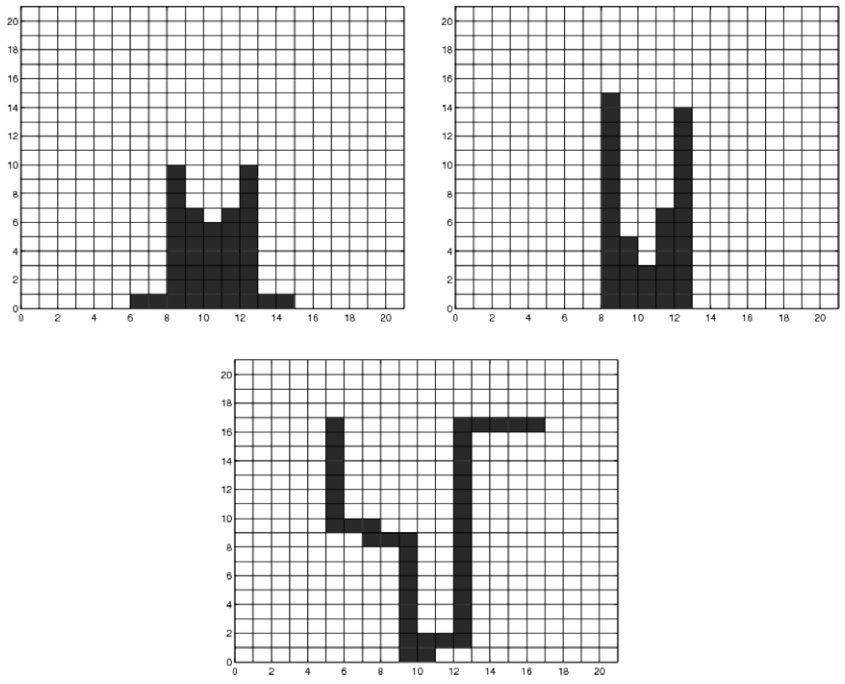

Fig. 3. Results by SA/GA for 21×21, $\hat{k} = 3, 10, 100$.

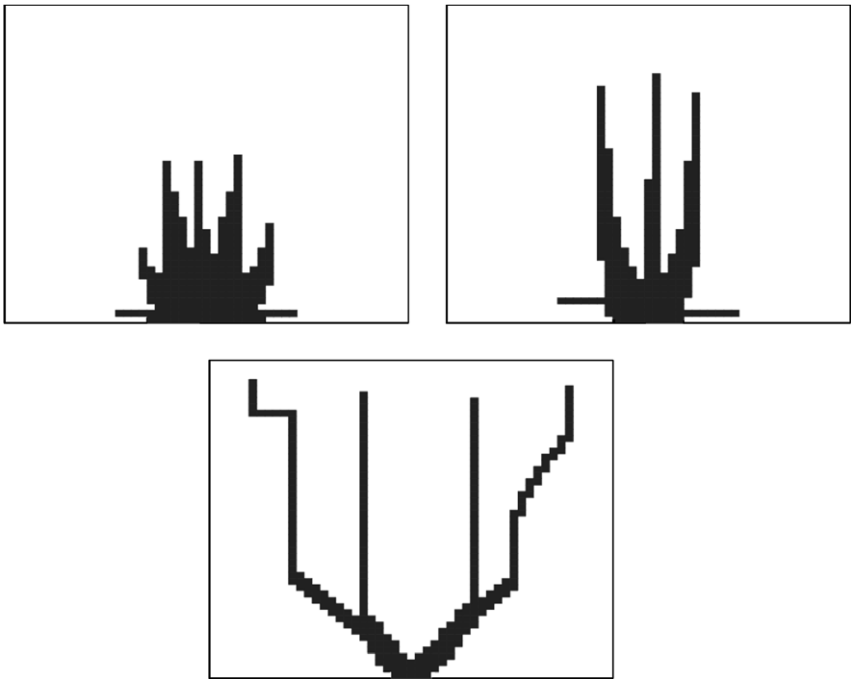

Fig. 4. Results by SA for 51×51, $\hat{k} = 3, 10, 100$.

The algorithms developed in this article can also be used to optimize the mean temperature of the volume, by changing the objective of Eq. (6) to min $\sum T_i$. The mean temperature optimized by SA is also lower than that by BO. Fig. 7 is the results by SA and BO for minimizing mean temperature with grid of 51×51, $\hat{k} = 100$. Mean temperature got by SA is 17.3% lower than that by BO. For other cases, results optimized by SA/GA are also better than that by BO, especially for high $\hat{k}$.

4.2. Compare with constructal theory

Comparison of combinatorial optimization algorithms and constructal theory is also taken. Suppose that

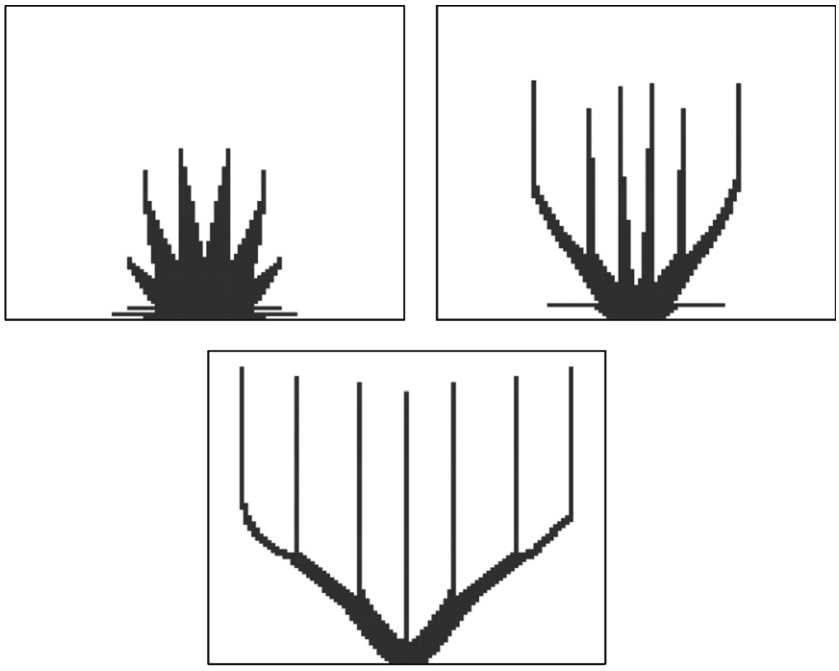

Fig. 5. Results by SA for 101×101, $\hat{k} = 3, 10, 100$.

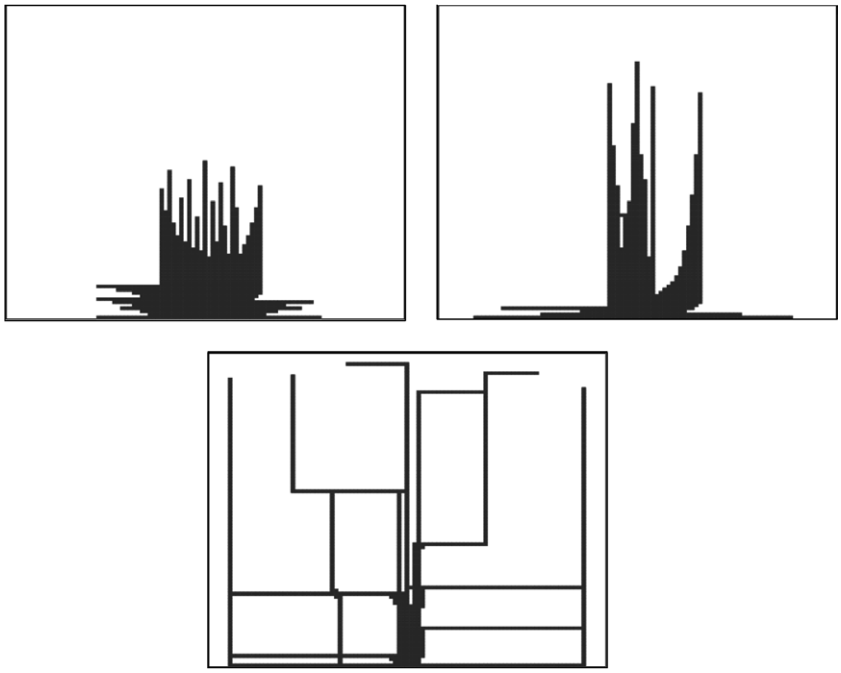

Fig. 6. Results by BO for 101×101, $\hat{k} = 3, 10, 100$.

Table 1
ρ got by SA/GA and BO for different grids and $\hat{k}$

Grid	21×21			51×51			101×101		
$\hat{k}$	3	10	100	3	10	100	3	10	100
SA/GA	0.7759	0.3915	0.1107	0.8268	0.4081	0.0792	0.9375	0.4244	0.0726
BO	0.7793	0.4010	0.1539	0.8690	0.4168	0.1091	0.9395	0.4385	0.0916
Diff. (%)	−0.4	−2.4	−28.1	−4.9	−2.1	−27.4	−0.2	−3.2	−20.7

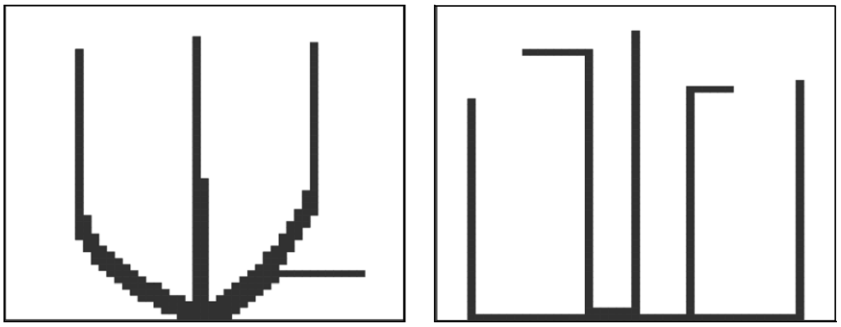

Fig. 7. Results by SA and BO for 51×51, $\hat{k} = 100$.

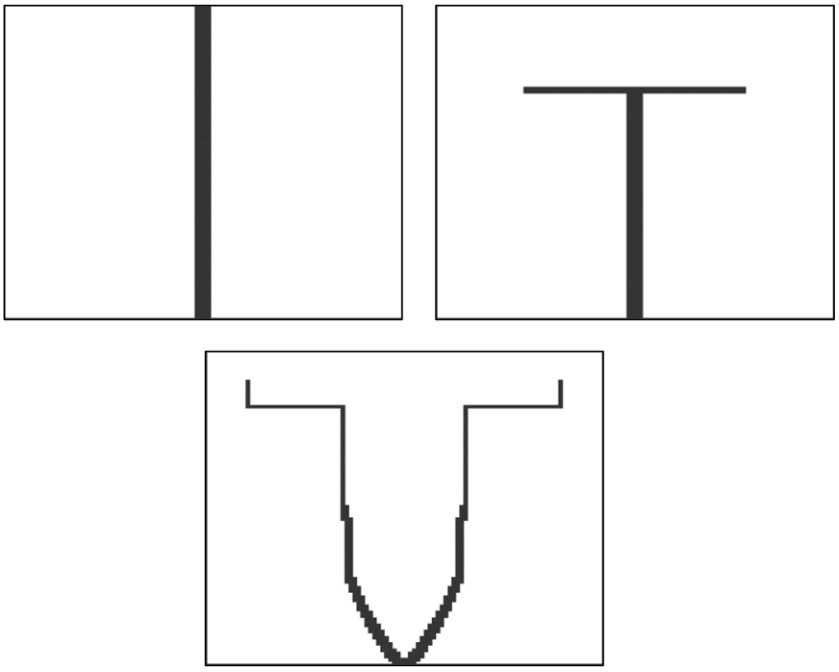

Fig. 8. Results by constructal theory and SA (50×50 and 100×100), $\hat{k} = 100$, $\phi = 0.04$.

$\hat{k} = 100$ and $\phi = 0.04$, according to Ghodoossi [9], the element area got by constructal theory (see first figure in Fig. 8) is better than high order assembly because $\hat{k}\phi = 4 < 8$, and the optimal shape is square because

$$\left(\frac{H}{L}\right)_{\text{opt}} = 2\sqrt{\frac{1}{\hat{k}\phi}} = 1 \tag{10}$$

The optimized nondimensional thermal resistance predicted by constructal theory is [1,9]

$$\rho = \frac{1}{2}\sqrt{\frac{1}{\hat{k}\phi}} = 0.25 \tag{11}$$

The nondimensional thermal resistance of element area calculated by numerical method is 0.2083. The difference between numerical and theoretical results is caused by the simplification of constructal theory. The nondimensional thermal resistance for the same problem optimized by SA using grid of 50×50 is 0.1855, about 10.9% lower than that by constructal theory, and it is 0.1517 using grid of 100×100, about 27.2% lower. The shapes got by constructal theory and SA are shown in Fig. 8.

For another case, suppose that $\hat{k} = 400$ and $\phi = 0.04$, so the first order assembly construct is the best shape by constructal theory because $\hat{k}\phi = 16 > 8$, and the optimal number of constituents $n = \sqrt{\hat{k}\phi} = 4$, optimal ratio of length and width is $L/W = \sqrt{2}$, the porosity of element level is $\phi_0 = \phi/2 = 0.02$ [9]. The optimal structure is shown in Fig. 9, in which the heat sink is located on the middle of bottom border, 2% of border width. The theoretical optimized nondimensional thermal resistance is [9]

$$\rho = \frac{\sqrt{2}}{\hat{k}\phi} = 0.0884 \tag{12}$$

while the numerical result is 0.0751. The nondimensional thermal resistance got by SA using grid of 100×100 is 0.0595, 20.8% lower than that by constructal theory. The shapes got by constructal theory and SA are shown in Fig. 9.

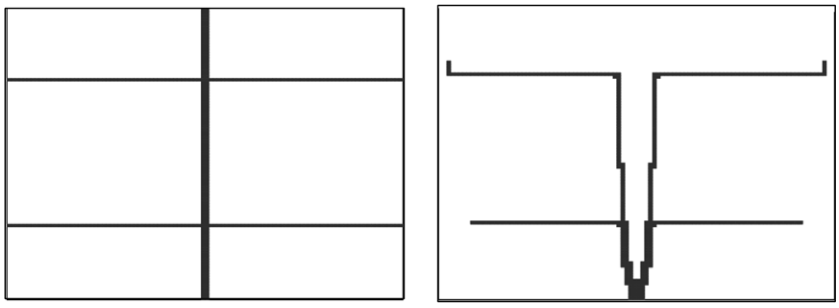

Fig. 9. Results by constructal theory and SA (100 × 100), $\hat{k} = 400$, $\phi = 0.04$.

According to these shapes, the shapes got by SA are better than that by constructal theory because the high conductivity material is more close to the corner.

5. Conclusion

In this article, a new method is developed to solve VP problem, which transforms the original nonlinear optimization problem to a combinatorial optimization problem, then uses combinatorial optimization algorithms to search the best combination of high conductivity material. This method can find the best solution theoretically, but due to limitations of computing capability only approximate best solution can be got. Comparing to bionic optimization algorithm and constructal theory through some cases, performance of the new method is better than them, especially for high conductivity ratio of high conductivity material and base material.

The shapes got by the new method and bionic optimization algorithm are similar, but more complicated than that by constructal theory. The shapes of high conductivity material change with the value of $\hat{k}$: when $\hat{k}$ is low, the shape is thick and short; when $\hat{k}$ is high, the shape becomes slender. The shapes also change with grid number of the volume. The complexity and performance increase as grid number increases, especially for high $\hat{k}$. The grid number is limited by manufacturing restrictions in practice. The computing time of the new method is highly correlated with grid number. For big grid number, parallel computing should be applied to SA and GA with high performance [20].

The new method can be easily generalized for more complex VP problems, such as nonuniform heat source, irregular shape, which are difficult for constructal theory. It can also be used for flexible objective. Besides minimization highest temperature and mean temperature of the whole volume, it can also optimize the temperature of a sub-volume inside of the volume, that is difficult for the other methods.

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